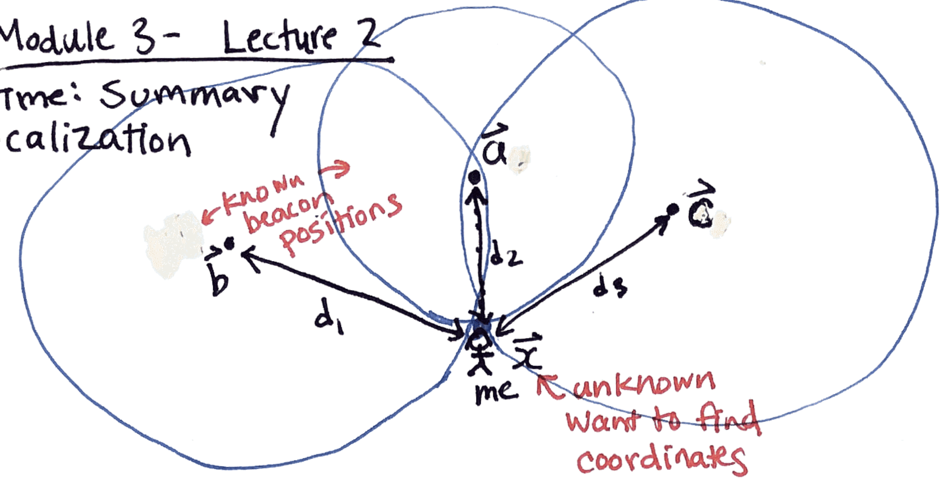
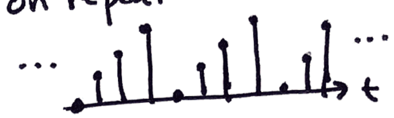
Module 3 - Lecture 2

Last time: Summary
Localization



Each beacon plays a song on repeat:



shifts @ reciever (me) due to dist: $d = vt$

How to find distances d_1, d_2, d_3?

For just ONE beacon:

song
N-periodic

$$\vec{s} = \begin{bmatrix} s(0) \\ s(1) \\ s(2) \\ \vdots \\ s(n-1) \end{bmatrix}_{N\times 1} \xrightarrow{\text{delay}=1} \vec{s}^{(1)} = \begin{bmatrix} s(n-1) \\ s(0) \\ s(1) \\ \vdots \\ s(n-2) \end{bmatrix}$$

"circular shift"

for a general delay of i (index of shift)

$$\vec{s}^{(i)} = \begin{bmatrix} s(n-i) \\ s(n-i+1) \\ \vdots \\ s(n-1) \\ s(0) \\ s(1) \\ \vdots \\ s(n-i-1) \end{bmatrix}$$

To find the shift of the recieved signal at me, we look at the inner product between $\vec{r}$ and $\vec{s}_i$ for all possible shifts.

inner product $\langle \vec{a}, \vec{b} \rangle = \vec{a}^T \vec{b}$

i.e. Find i that maximizes $\vec{s}^{(i)T} \vec{r} \rightarrow$ correct shift

To compute the inner products in an organized way, put them into a vector:

inn. prod. w shift = 0

inn. prod. w shift = n-1

$$\begin{bmatrix} \vec{s}^{(0)T} \vec{r} \\ \vec{s}^{(1)T} \vec{r} \\ \vec{s}^{(2)T} \vec{r} \\ \vdots \\ \vec{s}^{(n-1)T} \vec{r} \end{bmatrix}_{N\times 1} = \underbrace{\boxed{\begin{bmatrix} \text{---} & \vec{s}^{(0)T} & \text{---} \\ \text{---} & \vec{s}^{(1)T} & \text{---} \\ \text{---} & \vec{s}^{(2)T} & \text{---} \\ & \vdots & \\ \text{---} & \vec{s}^{(n-1)T} & \text{---} \end{bmatrix}}}_{N\times N} \vec{r}_{N\times 1} = \underbrace{C_{\vec{s}}}_{\text{correlates } \vec{s} \text{ with } \vec{r}} \vec{r}$$

$\equiv C_{\vec{s}}$

each entry is a scalar. Look for largest one → shift.

Looks like a mtx-vector multiplication in "Row view"

$$C_{\vec{s}} = \begin{bmatrix} s(0) & s(1) & s(2) & \cdots & s(n-1) \\ s(n-1) & s(0) & s(1) & \cdots & s(n-2) \\ s(n-2) & s(n-1) & s(0) & \cdots & s(n-3) \\ \vdots & & \ddots & & \vdots \\ s(1) & s(2) & s(3) & \cdots & s(0) \end{bmatrix}$$

NxN matrix that is fully defined by Nx1 vector!

Notice patterns!
↳ each row/col is circularly shifted version of prev.
"CIRCULANT MATRIX"

In general, we write correlation as: $\underbrace{\vec{\rho}_{\vec{x}\vec{y}}}_{\text{"correlation of } \vec{x} \,\&\, \vec{y}\text{"}} = C_{\vec{y}}\,\vec{x}$

cross

make $\vec{y}$ into $N \times N$ circulant matrix

$\boxed{\text{What is } \vec{\rho}_{\vec{x}\vec{x}}?}$ "auto-correlation"

Example:

$$\vec{x} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \qquad C_{\vec{x}} = \begin{bmatrix} 1 & 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 \end{bmatrix} \qquad \vec{\rho}_{\vec{x}\vec{x}} = C_{\vec{x}}\vec{x} = \begin{bmatrix} 5 \\ -3 \\ 1 \\ 1 \\ -3 \end{bmatrix}$$

1st entry Largest! always

$\boxed{\text{Why?}}$ (square of elements)

2D Example:

$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \qquad C_{\vec{a}} = \begin{bmatrix} a_1 & a_2 \\ a_2 & a_1 \end{bmatrix} \qquad \vec{\rho}_{\vec{a}\vec{a}} = \begin{bmatrix} a_1 & a_2 \\ a_2 & a_1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_1^2 + a_2^2 \\ 2a_1a_2 \end{bmatrix}$$

$\boxed{\text{Is } a_1^2 + a_2^2 \geq 2a_1a_2 \text{ ALWAYS?}}$

* When you match up elements they square to give large sum when most 'similar'

Yes! $a_1^2 + a_2^2 - 2a_1a_2 = (a_1 - a_2)^2 \geq 0$

$\Rightarrow a_1^2 + a_2^2 \geq 2a_1a_2$

'Geometric' interpretation:

check inn. prod. for shift m

$$\vec{\rho}_{\vec{x}\vec{x}}[m] = \vec{x}^{(m)T}\vec{x}$$

$$= \underbrace{\|\vec{x}^{(m)}\|\,\|\vec{x}\|}_{\text{length of vector doesn't change w/ shift}} \underbrace{\cos\theta}_{\text{larger when aligned!}}$$

zero shift $\Rightarrow$ Aligned!

Locationing Summary: $\boxed{\text{if } \vec{\rho}_{\vec{r}\vec{s}}[m] \text{ is the largest of all correlation values, then the delay of signal is } m}$

$\boxed{\text{Why is inner product the right metric?}}$ (to decide shift)

→ noise robust (works even when dog barks)

→ can handle attenuation (when song is quieter than expected)

→ will work well even when multiple beacons are singing (reciever gets lin. combo. of multiple songs)

$\boxed{\text{Could I have solved it with GE and inverse matrix?}}$

In single beacon case → Yes

Ex. (1 beacon) $\vec{r} = \vec{s}^{(m)} = \begin{bmatrix} | & | & & | \\ \vec{s}^{(0)} & \vec{s}^{(1)} & \cdots & \vec{s}^{(n-1)} \\ | & | & & | \end{bmatrix} \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\text{call it } \beta}$ ← m^{th} index

transpose of circulant matrix $C_{\vec{s}}$

$$\vec{r} = C_{\vec{s}}^T \vec{\beta}$$

↖ solve for me!

Could have used $\vec{\beta} = (C_{\vec{s}}^T)^{-1} \vec{r}$ to get shift index.

When are $C_{\vec{s}}^{-1}$ and $C_{\vec{s}}^T$ analogous? If song is orthogonal (or almost) to shifted versions of itself.

Multiple Beacons: Each beacon $\{a, b, c\}$ sings its song $\{\vec{s}_a, \vec{s}_b, \vec{s}_c\}$

Should songs be same? No, then can't tell which beacon

Recieved signal is ~~sum~~ sum of all three:

$$\vec{r} = \vec{s}_a^{(N_a)} + \vec{s}_b^{(N_b)} + \vec{s}_c^{(N_c)}$$

index of shift

index of beacon

What correlation do I do to find the song shift index for beacon a? (N_a)

cross-corr. recieved w/ beacon a

$$\vec{\rho}_{\vec{r}\vec{s}_a}[m] = \langle \vec{r}, \vec{s}_a^{(m)} \rangle$$

find m that maximizes this → N_a

$$= \vec{s}_a^{(m)T} \left(\vec{s}_a^{(N_a)} + \vec{s}_b^{(N_b)} + \vec{s}_c^{(N_c)} \right)$$

$$= \underbrace{\vec{s}_a^{(m)T} \vec{s}_a^{(N_a)}}_{\text{corr. } \vec{s}_a \text{ with itself (shifted)}} + \underbrace{\vec{s}_a^{(m)T} \vec{s}_b^{(N_b)}}_{\text{corr w/ } \vec{s}_b} + \underbrace{\vec{s}_a^{(m)T} \vec{s}_c^{(N_c)}}_{\text{corr. w/ } \vec{s}_c}$$

should be LARGE @ m ⇒ N_a

should be SMALL to not ruin 1st term

How can we make sure this finds shift?

Add new conditions:

1) $\vec{s}_a, \vec{s}_b, \vec{s}_c$ should be orthogonal (or almost) to each other, so their cross-corr. are small

2) Auto-corr. of $\vec{s}_a, \vec{s}_b, \vec{s}_c$ should be SMALL at all other shifts $\neq 0$

Ex. Is $\vec{s} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ a good song? No (auto-corr. high)

Is $\vec{s} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}$ a good song? No (repeats)

Is $\vec{s}_a = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\vec{s}_b = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ good? No.

Then what?

I found shift N_a. Now what?

Next, do $\vec{\rho}_{\vec{r}\vec{s}_b}[m]$ to find N_b

and $\vec{\rho}_{\vec{r}\vec{s}_c}[m]$ to find N_c

N_a, N_b, N_c represent delay in # samples.
Need to convert to actual time. $\boxed{\text{How?}}$

Ex. sampling frequency = 44,000 samples/sec → T_s = 1 sample = $\frac{1}{44{,}000}$ sec

time per sample

$$N_a \rightarrow t_a = N_a \cdot T_s = \frac{N_a}{44{,}000} \text{ seconds}$$

delay in time units

$\boxed{\text{Then what?}}$

~~Find~~ distance of me from beacon a:

$$d_a = v\, t_a$$

time of delay

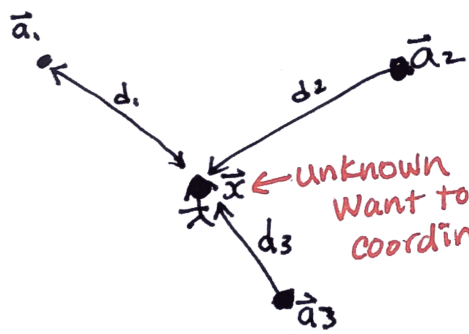
speed of signal

[Am I done?]

No. Now I need to find my location relative to known beacons. $\boxed{\text{How?}}$

I *could* draw circles, but how does computer do it?

$\boxed{\text{Trilateration}}$ how to get position from dist. measurements + beacon positions.



unknown
want to find coordinates.

we know:

① $d_1^2 = \|\vec{x} - \vec{a}_1\|^2 = \vec{x}^T\vec{x} - 2\vec{a}_1^T\vec{x} + \|a_1\|^2$

② $d_2^2 = \|\vec{x} - \vec{a}_2\|^2 = \vec{x}^T\vec{x} - 2\vec{a}_2^T\vec{x} + \|a_2\|^2$

③ $d_3^2 = \|\vec{x} - \vec{a}_3\|^2 = \underbrace{\vec{x}^T\vec{x} - 2\vec{a}_3^T\vec{x} + \|a_3\|^2}_{\text{'transpose notation'}}$

oh no! Nonlinear terms ☹

Trick: ②-① $2(\vec{a}_1 - \vec{a}_2)^T\vec{x} = \|a_1\|^2 - \|a_2\|^2 + d_2^2 - d_1^2$

③-① $2(\vec{a}_1 - \vec{a}_3)^T\vec{x} = \|a_1\|^2 - \|a_3\|^2 + d_3^2 - d_1^2$

put in matrix form

$$\begin{bmatrix} 2(\vec{a}_1 - \vec{a}_2)^T \\ 2(\vec{a}_1 - \vec{a}_3)^T \end{bmatrix} \vec{x} = \begin{bmatrix} \|a_1\|^2 - \|a_2\|^2 + d_2^2 - d_1^2 \\ \|a_1\|^2 - \|a_3\|^2 + d_3^2 - d_1^2 \end{bmatrix}$$

→ It's LINEAR in $\vec{x}$ now

Then, solve for x!

Does this say anything about where to place beacons?

Cannot be collinear!